APPENDIX 101: Fluid dynamics, Flow of gases and steam through nozzles, Derivation of equation for gas velocity at nozzle outlet

The outlet velocity from the nozzle can be derived from the equation of the first law of thermodynamics (e.g. [Škorpík, 2024]) for the case of flow of an ideal fluid not doing external work, not sharing heat with the surroundings and without significant effect of potential energy change:

$$\mathrm{d}h+\frac{\mathrm{d}V^2}{2}=0.$$

A101/2: The change in velocity and enthalpy between the inlet and any flow area of the nozzle, e.g. at the outlet, is obtained by integrating the last equation over the relevant section:

$$\int_{h\mathrm{i}}^{h\mathrm{e}}\mathrm{d}h=-\int_{V\mathrm{i}}^{V\mathrm{e}}\frac{\mathrm{d}V^2}{2}$$
$$h_\mathrm{e}-h_\mathrm{i}=\frac{V_\mathrm{i}^2}{2}-\frac{V_e^2}{2}$$
$$V_\mathrm{e}=\sqrt{2(h_\mathrm{i}-h_\mathrm{e})+V_\mathrm{i}^2}.$$

From the enthalpy definition formula [Škorpík, 2024] can also be written for the enthalpy difference:

$$h_\mathrm{i}-h_\mathrm{e}=c_\mathrm{p}(T_\mathrm{i}-T_\mathrm{e}),$$

c_p [J·K^{-1}·kg^{-1}] heat capacity of 1 kg of substance at constant pressure.

A101/3: From Mayer's relation [Kalčík and Sýkora, 1979] for the relationship between the ratio of heat capacities κ and the individual gas constant r, we can write for the heat capacity at constant volume:

$$c_\mathrm{p}=\frac{\kappa}{\kappa-1}r.$$

So the last equation can be adapted to the shape:

$$h_\mathrm{i}-h_\mathrm{e}=c_\mathrm{p}=\frac{\kappa}{\kappa-1}r(T_\mathrm{i}-T_\mathrm{e}).$$

A101/4: The relationship between the initial temperature T_i and the final temperature T_e can be described by the isentropy equation:

$$p_\mathrm{i}\cdot v_\mathrm{i}^{\kappa}=p_\mathrm{e}\cdot v_\mathrm{e}^{\kappa}.$$

The relationship between specific volume v and temperature T can be expressed from the ideal gas equation of state:

$$p\cdot v=r\cdot T\Rightarrow v=\frac{r\cdot T}{p}.$$

Combining the last two equations, we obtain the relationship for the temperature T_e:

$$p_i \frac{r \cdot T_i^{\kappa}}{p_i} = p_e \frac{r \cdot T_e^{\kappa}}{p_e} \Rightarrow T_e = T_i \left(\frac{p_i}{p_e} \right)^{\frac{1-\kappa}{\kappa}}.$$

A101/5: Substituting the temperature T_e into the equation in A101/3 and modifying it, we obtain the relationship for the enthalpy difference:

$$h_i - h_e = \frac{\kappa}{\kappa - 1} r \cdot T_i \left[1 - \left(\frac{p_e}{p_i} \right)^{\frac{\kappa-1}{\kappa}} \right].$$

A101/6: Substitute the last equation into the gas velocity equation V_e in A101/1:

$$V_e = \sqrt{\frac{2 \cdot \kappa}{\kappa - 1} r \cdot T_i \left[1 - \left(\frac{p_e}{p_i} \right)^{\frac{\kappa-1}{\kappa}} \right] + V_i^2}.$$

A101/7: The calculations are often based on the stagnation state of the gas upstream of the nozzle, i.e. as if the gas in the nozzle were expanding from the point h_s in FIGURE 101 (p. 4), then the equation for the velocity can be derived in the form:

$$V_e = \sqrt{2(h_{is} - h_e)} = \sqrt{\frac{2\kappa}{\kappa - 1} r \cdot T_{is} \left(1 - \varepsilon_s^{\frac{\kappa-1}{\kappa}} \right)}.$$